

Mass of the charm-quark from QCD sum rules

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Abstract

Relativistic and non-relativistic ratios of Laplace transform QCD moment sum rules for charmonium are used in order to determine the value of the on-shell charm-quark mass. The validity of the non-relativistic version of QCD sum rules in this particular application is discussed. After using current values of the perturbative and non-perturbative QCD parameters, as well as experimental data on the J/ψ system, we obtain $m_c(Q^2 = m_c^2) = 1.46 \pm 0.07$ GeV.

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Recently, the value of the (on-shell) beauty-quark mass has been determined [1] by confronting very accurate experimental data on the upsilon system [2] with ratios of non-relativistic Laplace transform QCD moments. This theoretical framework, suggested by Bertlmann [3], offers several advantages, e.g. radiative and non-perturbative corrections are well under control, and the non-relativistic limit follows quite naturally from quantum mechanical analogues [4]. This version of QCD sum rules leads to an expansion in powers of the inverse of the heavy quark mass which allows one to test the range of validity of the non-relativistic limit, and more generally, to assess the role of mass corrections. This might be of interest for calculations based on the simplifying assumption $\Lambda_{QCD}/m_Q \ll 1$.

Non-relativistic Laplace moments appear to have a sensitive dependence on the quark mass. In fact, in spite of the large uncertainties affecting the values of Λ_{QCD} and the non-perturbative gluon condensate, m_b can be extracted from the upsilon data with high precision. This extraction is performed by confronting the ratios of Laplace transform moments calculated from experiment with those from theory. The latter involve the QCD parameters m_b , Λ , $\langle \alpha_s G^2 \rangle$, etc.. These ratios are functions of the Laplace variable, which acts as a short distance expansion parameter, and one finds a reasonably wide region in this variable where there is a matching between experiment and theory for a specific value of the quark mass.

As pointed out in [1], a straightforward extension of this technique to the charm-quark may not work, as radiative and mass corrections could exceed 100 % . This would be true if the window in the Laplace variable would be the same for beauty and for charm. However, there is no *a-priori* reason for this to be the case. In fact, as also suggested in [3], the matching between theory and experiment for the beauty and the charm quarks could take place at different ranges of the Laplace variable. If this range is such that radiative, non-perturbative, and mass corrections remain small, then it would become possible to extract the value of the charm quark from this framework.

In this note we study the ratios of relativistic Laplace transform QCD moments for the case of charm, and compare them with the non-relativistic versions. This provides a measure of the validity of the heavy quark mass expansion in this particular application, as well as an estimate of the systematic uncertainties affecting this technique. We make use of the current values of Λ and $\langle \alpha_s G^2 \rangle$, and carry out the non-relativistic expansion of the Laplace ratios to next-to-next to leading order in $1/m_c$. We find this expansion to converge reasonably fast, and the radiative and non-perturbative contributions to be safely under control. Thanks to this feature and to the high accuracy of the experimental data, as in the case of m_b , the extracted value of the (on-shell) charm-quark mass is affected by a relatively small uncertainty, in spite of the large uncertainties in Λ and $\langle \alpha_s G^2 \rangle$. In connection with the latter, we recall that most of the early (relativistic) QCD sum rule analyses of charmonium attempted to extract the values of both the charm-quark mass and the gluon condensate. Since $\langle \alpha_s G^2 \rangle$ is now known independently from e^+e^- and from τ decay data [5], one is in a better position to determine m_c . In fact, for a given pair of values of Λ and $\langle \alpha_s G^2 \rangle$, the matching between theory and experiment becomes a one parameter fit, the parameter being m_c . We find that the values of m_c from the fully relativistic ratios are in good agreement (within errors) with those from the non-relativistic ratios. We conclude with a comparison of our results with those obtained previously by other authors.

We begin by considering the two-point function

$$\Pi_{\mu\nu}(q) = i \int d^4x \exp(iqx) \langle 0 | T(V_\mu(x) V_\nu^+(0)) | 0 \rangle = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi(q^2), \quad (1)$$

with $V_\mu(x) = \bar{c}(x) \gamma_\mu c(x)$. The function $\Pi(q^2)$ has been calculated in perturbative QCD at the two-loop level [6], with its imaginary part given by

$$\frac{1}{\pi} \text{Im } \Pi(s)|_{QCD} = \frac{1}{8\pi^2} v(3-v^2) \left\{ 1 + \frac{4\alpha_s}{3} \left[\frac{\pi}{2v} - \frac{(v+3)}{4} \left(\frac{\pi}{2} - \frac{3}{4\pi} \right) \right] \right\} \theta(s - 4m_c^2), \quad (2)$$

where $v = \sqrt{1 - 4m_c^2/s}$, and m_c is the charm-quark on-shell mass: $m_c = m_c(Q^2 = m_c^2)$.

The leading non-perturbative term in the operator product expansion of $\Pi(q^2)$ involves the gluon condensate, i.e.

$$\Pi(s)|_{NP} = \frac{1}{48 s^2} \times \left[\frac{3(v^2 + 1)(1 - v^2)^2}{2v^5} \ln \frac{1 + v}{1 - v} - \frac{3v^4 - 2v^2 + 3}{v^4} \right] \langle \frac{\alpha_s}{\pi} G^2 \rangle . \quad (3)$$

The function $\Pi(q^2)$ satisfies a once-subtracted dispersion relation, and the subtraction constant can be disposed of by taking the Laplace transform

$$\Pi(\sigma) = \int_0^\infty ds \exp(-\sigma s) \text{Im } \Pi(s) . \quad (4)$$

The quantity of interest to us here is the ratio of the first two Laplace moments, which can be expressed as

$$\mathcal{R}(\sigma) = -\frac{d}{d\sigma} \ln \Pi(\sigma) . \quad (5)$$

From (2) and (3) one obtains [3]

$$\Pi(\sigma) = \exp(-4m_c^2\sigma)\pi A(\sigma)[1 + a(\sigma)\alpha_s + b(\sigma)\phi] , \quad (6)$$

where, with $\omega = 4m_c^2\sigma$,

$$\pi A(\omega) = \frac{3}{16\sqrt{\pi}} \frac{4m_c^2}{\omega} G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) , \quad (7)$$

$$a(\omega) = \frac{4}{3\sqrt{\pi}} G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) [\pi - c_1 G(1, 2, \omega) + \frac{1}{3}c_2 G(2, 3, \omega)] - c_2 , \quad (8)$$

$$b(\omega) = -\frac{\omega^2}{2} G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) , \quad (9)$$

$$c_1 = \frac{\pi}{3} + \frac{c_2}{2} , \quad c_2 = \frac{\pi}{2} - \frac{3}{4\pi} , \quad (10)$$

$$\alpha_s(Q^2) = \frac{12\pi}{25 \ln Q^2/\Lambda^2} , \quad (11)$$

$$\phi = \frac{\pi}{36} \frac{\langle \alpha_s G^2 \rangle}{m_c^4} , \quad (12)$$

$$G(b, c, \omega) = \frac{\omega^{-b}}{\Gamma(c)} \int_0^\infty dt t^{c-1} e^{-t} \left(1 + \frac{t}{\omega}\right)^{-b} . \quad (13)$$

The function $G(b, c, \omega)$ is related to the Whittaker function $W_{\lambda, \mu}(\omega)$ through [7]

$$G(b, c, \omega) = \omega^{\mu-1/2} e^{\omega/2} W_{\lambda, \mu}(\omega) , \quad (14)$$

with $\mu = (c - b)/2$, and $\lambda = (1 - c - b)/2$. The ratio (5) can then be calculated, with the result [3]

$$\mathcal{R}(\omega) = 4m_c^2 \left[1 - \frac{A'(\omega)}{A(\omega)} - \frac{a'(\omega)\alpha_s + b'(\omega)\phi}{1 + a(\omega)\alpha_s + b(\omega)\phi} \right] , \quad (15)$$

where

$$A'(\omega) = -\frac{A(\omega)}{\omega} \left[\frac{3}{2} - \frac{5}{4} G\left(\frac{3}{2}, \frac{7}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \right] , \quad (16)$$

$$\begin{aligned} a'(\omega) = & \frac{4}{3\omega\sqrt{\pi}} G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left\{ \frac{1}{2} G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left[G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \right. \right. \\ & \left. \left. - \frac{5}{2} G\left(\frac{3}{2}, \frac{7}{2}, \omega\right) \right] \left[\pi - c_1 G(1, 2, \omega) + \frac{c_2}{3} G(2, 3, \omega) \right] \right. \\ & \left. + c_1 [G(1, 2, \omega) - 2G(2, 3, \omega)] + \frac{1}{3} c_2 [-2G(2, 3, \omega) + 6G(3, 4, \omega)] \right\} , \end{aligned} \quad (17)$$

$$b'(\omega) = \frac{2}{\omega}b(\omega) - \frac{\omega}{4} \left\{ G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) - \frac{3}{2} + G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) G^{-2}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left[G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) - \frac{5}{2} G\left(\frac{3}{2}, \frac{7}{2}, \omega\right) \right] \right\} . \quad (18)$$

The above expressions (15)-(18) involve no approximations, other than the two-loop perturbative expansion, and retaining the leading non-perturbative term in the operator product expansion. We shall refer to Eq.(15) as the fully relativistic Laplace ratio.

In the non-relativistic (heavy quark-mass) limit, the Laplace transform (4) becomes

$$\Pi(\tau) = \int_0^\infty dE \exp(-\tau E) \text{Im } \Pi(E) , \quad (19)$$

where $\tau = 4m_c\sigma$, and $s = (2m_c + E)^2$ so that $E \geq 0$. The Laplace ratio (5) is now given by

$$\mathcal{R}(\tau) = 2m_c - \frac{d}{d\tau} \ln \Pi(\tau) . \quad (20)$$

After expanding the functions $G(b, c, \omega)$ entering Eqs.(7)-(9), and (16)-(18), we obtain the non-relativistic ratio

$$\begin{aligned} \mathcal{R}(\tau) = 2m_c & \left\{ 1 + \frac{3}{4} \frac{1}{m_c\tau} \left(1 - \frac{5}{6} \frac{1}{m_c\tau} + \frac{10}{3} \frac{1}{m_c^2\tau^2} \right) \right. \\ & - \frac{\sqrt{\pi}}{3} \frac{\alpha_s}{\sqrt{m_c\tau}} \left[1 - \left(\frac{2}{3} + \frac{3}{8\pi^2} \right) \frac{1}{m_c\tau} + \frac{1}{32} \left(107 + \frac{51}{\pi^2} \right) \frac{1}{m_c^2\tau^2} \right] \\ & \left. + \frac{\pi}{48} \frac{\tau^2}{m_c^2} \langle \alpha_s G^2 \rangle \left(1 + \frac{4}{3} \frac{1}{m_c\tau} - \frac{5}{12} \frac{1}{m_c^2\tau^2} \right) \right\} . \quad (21) \end{aligned}$$

The appearance of $\sqrt{m_c}$ above is only an artifact of the change of variables; written in terms of σ , Eq.(21) contains no such term. The theoretical ratios of the first two Laplace moments (15) and (21) must now be confronted with a corresponding ratio involving the experimental data on the J/ψ system. We parametrize the latter by a sum of two

narrow resonances below $D\bar{D}$ threshold, followed by a hadronic continuum modelled by perturbative QCD, which gives

$$\Pi(\sigma)|_{EXP} = \frac{27}{16\pi} \frac{1}{\alpha_{EM}^2} \sum_V \Gamma_V^{ee} M_V \exp(-\sigma M_V^2) + \frac{1}{\pi} \int_{s_0}^{\infty} ds \exp(-\sigma s) \text{Im } \Pi(s)|_{QCD} . \quad (22)$$

The experimental ratio is then calculated using (22) in (5). The continuum threshold s_0 is chosen at or below the $D\bar{D}$ threshold. Reasonable changes in the value of s_0 have essentially no impact on the results, as $\Pi(\sigma)$ is saturated almost entirely by the first two J/ψ narrow resonances.

In the theoretical ratios we use the current values: $\Lambda = 200 - 300$ MeV , for four flavours [2], and $\langle \alpha_s G^2 \rangle = 0.063 - 0.19$ GeV⁴ [5]. We find that theoretical and experimental ratios match in the wide sum rule window: $\sigma \simeq 0.8 - 1.5$ GeV⁻², for $m_c = 1.39 - 1.46$ GeV in the fully relativistic case; and $\sigma \simeq 0.6 - 0.8$ GeV⁻², $m_c = 1.40 - 1.53$ GeV in the non-relativistic case. Figure 1 shows the behaviour of (15) for $\Lambda = 200$ MeV, and $\langle \alpha_s G^2 \rangle = 0.063$ GeV⁴ (solid curve), together with the experimental ratio calculated from (22) (broken curve), corresponding to $m_c = 1.44$ GeV. A similar qualitative behaviour is obtained for other values of Λ and the gluon condensate, both in the relativistic and the non-relativistic versions of the sum rules. For values of σ inside the sum rule window, the hierarchy of the various terms in the non-relativistic Laplace ratio (21) guarantees a fast convergence. In fact, the leading correction in $1/m_c$ is at the 15-20% level, the radiative correction and the non-perturbative contribution amount both to less than 10% . At the same time, the next, and next-to-next to leading (in $1/m_c$) terms everywhere in (21) are safely small, as it can be easily verified from (21) noticing that if $\sigma \simeq 1/2$ GeV⁻², then $\tau \simeq 2m_c$. Clearly, the complete analysis at the level of accuracy of these next-to-leading mass corrections would require the evaluation of the perturbative $O(\alpha_s^2)$ terms. Combining the results from both versions of the Laplace ratios, leads to the

result

$$m_c(Q^2 = m_c^2) = 1.46 \pm 0.07 \text{ GeV} . \quad (23)$$

In order to facilitate the comparison of (23) with previous determinations based on various versions of QCD sum rules [3], [8] - [14], we show in Fig. (2) the dependence of m_c on Λ for three different values of $\langle \alpha_s G^2 \rangle$, and in Fig. (3) the dependence of m_c on the gluon condensate for Λ in the range: $\Lambda = 100 - 400 \text{ MeV}$. Both figures correspond to the fully relativistic version of the QCD sum rules. Figures 4 and 5 refer to the non-relativistic determination. In comparing values of m_c from different determinations, it is important to know which values of Λ and $\langle \alpha_s G^2 \rangle$ have been used, as well as which renormalization point has been chosen, e.g. some authors determine $m_c(Q^2 = -m_c^2)$, which is related to the on-shell mass $m_c(m_c^2)$ through

$$m_c^2(m_c^2) = m_c^2(-m_c^2) \left(1 + \frac{4 \ln 2}{\pi} \alpha_s \right) . \quad (24)$$

After using the same values of Λ and $\langle \alpha_s G^2 \rangle$ as used in [3], [8]-[14], and after converting to the on-shell mass (if necessary), we find that results from the present method are in very good agreement with those of [3] and [8], and agree within errors with [9]-[11]; the latter determinations being on the low side of our error bars. On the other hand, the technique used here gives values of m_c somewhat higher than those obtained in [12]-[14].

Recently [15], the two-loop correction to the Wilson coefficient of the gluon condensate has been calculated. We have incorporated this additional term in the Laplace ratios, and find that the correction it introduces is about a factor of 2 larger than the $1/m_c$ correction to $\langle \alpha_s G^2 \rangle$ but with an opposite sign. Given the relative smallness of the overall contribution from the gluon condensate, and the conservatively large uncertainty we have allowed in its value, the final result for m_c remains basically unchanged.

Finally, regarding the recent observation that the definition of heavy quark pole mass in the context of the Heavy Quark Effective Theory (HQET) should contain some intrinsic

ambiguity beyond perturbation theory [16], due to nonperturbative long distance effects, we notice that the hadronic system considered here is made of two heavy quarks, while HQET strictly applies to heavy-light bound states.² The non-relativistic ratio (21) is not a relation of the HQET, although it is obtained formally from the relativistic ratio (15) in the large quark-mass limit. The purpose of considering (21) together with (15) has been to assess the size of relativistic corrections, as well as of systematic uncertainties of QCD sum rules. The numerical results, and in particular the consistency between the relativistic and nonrelativistic determinations of m_c , indicate that these effects should be small.

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Figure Captions

Figure 1: The fully relativistic ratio (15) (solid curve), and the experimental ratio (broken curve). The values of the parameters are: $\Lambda = 200$ MeV, $\langle \alpha_s G^2 \rangle = 0.063$ GeV⁴, and $m_c = 1.44$ GeV.

Figure 2: Dependence of m_c on Λ for $\langle\alpha_s G^2\rangle = 0.038 \text{ GeV}^4$ (curve (a)), 0.063 GeV^4 (curve (b)), and 0.19 GeV^4 (curve (c)). The fully relativistic ratio (15) has been used. Curve (a) is shown for reference purposes, as this low value of $\langle\alpha_s G^2\rangle$ was not used in our analysis.

Figure 3: Dependence of m_c on $\langle\alpha_s G^2\rangle$ for $\Lambda = 100 \text{ MeV}$ (curve (a)), 200 MeV (curve (b)), 300 MeV (curve (c)), and 400 MeV (curve (d)). The fully relativistic ratio (15) has been used. Curves (a) and (d) are shown for reference purposes, as these extreme values of Λ were not used in our analysis.

Figure 4: Same as in Fig.2, except that the non-relativistic ratio (21) has been used.

Figure 5: Same as in Fig.3, except that the non-relativistic ratio (21) has been used.

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